The Tidal Motion in the Irish Sea, its Currents and its Energy.

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The first part of this paper is devoted to the derivation of results from the Laplacian theory, connecting the tidal flow and the form of the free surface of a sea rotating with the earth, and small enough to be taken as flat. The work is carried to a first approximation only, the motion relative to the earth being considered small in magnitude and simply harmonic with respect to the time. The relations obtained have been applied numerically to the tides in the Irish Sea, chosen because of the comparatively large amount of data available, and the agreement between theory and observation seems to be satisfactory.

In the second portion of the paper, the hydro-dynamical equations are discussed to a further approximation, and the mean rates of flow of water and of transfer of energy across vertical sections are deduced, both depending on terms of the second order. These quantities are evaluated in the case of the Irish Sea, and two important results are obtained. It is shown that there is a steady residual drift of water northwards, of such a magnitude that the Sea would empty itself through the North Channel about once a year. The flow of energy into the Irish Sea through the two channels is also examined, and the total mean rate is found to be of the order 6×10^{17} ergs per second. This is approximately the same as the result obtained recently by Mr. G. I. Taylor,* working on the assumption of broken hydraulic flow, and confirmed, by calculation of the flow of energy, on the hypothesis of the current being a simple harmonic function of the time, and the same at all points in a vertical line.

The assumption of a purely harmonic flow clearly involves that the uncompensated part of the transfer of both water and energy must take place within a distance from the mean surface level equal to the amplitude in elevation of the tidal wave. On the other hand, the presence of a non-harmonic part in the flow may cause such transfer at all levels. It is shown that, in the case of the residual flow of water in the Irish Sea, the latter part is of importance in the result.

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* G. I. Taylor, 'Phil. Trans.,' A, vol. 220, pp. 1–33 (1919). Calculations based on this latter assumption were made by the present writer in 1916 in connection with his previous paper ('Roy. Soc. Proc.,' A, vol. 93, pp. 348–359 (1917)), but were held up on account of the uncertainty of the effect both of the non-periodic terms in the current flow and also of the variation of this flow with the depth. The completion of the work has been prevented until the present time by teaching duties and military service.

The total mean rate of flow of tidal energy into the Irish Sea thus determined is about 250 times the mean rate of dissipation of energy in the deeper portion of this region deduced by the author from a result involving surface velocities only, calculated on the tacit assumption of laminar motion throughout. The tidal energy is that of motion relative to the rotating earth, but it can easily be seen that this relative energy is independently conserved, although the total energy of the northward-moving stream of water varies with the latitude. Thus, any disappearance of the relative energy must take place *in situ* and, as Mr. Taylor contends, is to be ascribed mainly to local turbulent or eddying motion.

The Form of the Tidal Wave determines the Surface Currents.

As a first approximation, we regard a portion of the sea as a flat area rotating about a vertical axis, Oz, with the component angular velocity of diurnal motion, and suppose that the relative motion of the water is very small. Let the axes of x, y rotate in their plane with the prescribed angular velocity, ω, and let u, v be the components of the horizontal velocity, relative to these axes, of the particle of water which at time t is at (x, y, z). The equations of motion are then*

$$\frac{\partial u}{\partial t}-2\omega v = -g\frac{\partial \zeta}{\partial x}, \qquad \frac{\partial v}{\partial t}+2\omega u = -g\frac{\partial \zeta}{\partial y}, \tag{1}$$

where ζ is the elevation of the surface above its mean level.

For a simple harmonic wave we can write

$$\zeta = \mathrm{H}\cos\sigma(t-\mathrm{T}), \tag{2}$$

so that, from (1), u, v are each of the form $a\sin\sigma t+b\cos\sigma t$. The hodograph of the (u, v) motion is thus an ellipse, which actually is very elongated, and by taking the x-axis along its major axis we can write

$$u = \mathrm{W}\cos\sigma(t-\tau), \qquad v = w\sin\sigma(t-\tau). \tag{3}$$

In equations (2), (3), H, T, W, w, τ are functions of x, y; W is the maximum, and w the minimum current velocity at the point, w/W being small numerically, T is the time of high water, and τ is the time of "flood-stream."

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* See for example Lamb, 'Hydrodynamics,' §206.

Substituting for ζ, u, v in (1) and remembering that the equations are true for all values of t we find

$$\left.\begin{aligned} gH_x &= (\sigma W + 2\omega w)\sin\sigma(T-\tau) \\ -gH_y &= (2\omega W + \sigma w)\cos\sigma(T-\tau) \\ g\sigma H T_x &= (\sigma W + 2\omega w)\cos\sigma(T-\tau) \\ g\sigma H T_y &= (2\omega W + \sigma w)\sin\sigma(T-\tau) \end{aligned}\right\}, \tag{4}$$

where the suffixes denote partial differentiations.

Now let $H' = \sqrt{(H_x^2 + H_y^2)}$, $T' = \sqrt{(T_x^2 + T_y^2)}$ so that H′, T′ are the maximum surface up-gradients of H, T: also let χ be the angle between the directions of H′, T′, measured in this sense. Then from equations (4) we obtain the equations

$$(\sigma \pm 2\omega)(W \pm w) = g\sqrt{\{H'^2 \pm 2\sigma H H' T' \sin\chi + \sigma^2 H^2 T'^2\}}. \tag{5}$$

These equations are independent of the directions of the axes; and if H, T are known everywhere in the neighbourhood of a point, we can calculate from them W, w, so that *the magnitudes of the surface currents can be deduced from the lines of equal range of the tide and the co-tidal lines.*

The elimination of T from (4) gives

$$\left(\frac{gH_x}{\sigma W + 2\omega w}\right)^2 + \left(\frac{gH_y}{2\omega W + \sigma w}\right)^2 = 1, \tag{6}$$

from which w can be calculated if W and the gradients of H are known.

For the present problem the equation of continuity is

$$-\frac{\partial\zeta}{\partial t} = \frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y},$$

where h is the mean depth. Substituting from (2) and (3) for ζ, u, v, and neglecting the variation in h, this gives for a region in which τ is constant

$$\frac{\partial W}{\partial x} = -\frac{\sigma H}{h}\sin\sigma(T-\tau), \qquad \frac{\partial w}{\partial y} = \frac{\sigma H}{h}\cos\sigma(T-\tau). \tag{7}$$

The Tides of the Irish Sea.

The main tidal wave from the Atlantic approaches the British Isles from a south-westerly direction, the crest being approximately at right angles to the direction of propagation. Ultimately it is divided into three distinct portions, which advance up the English Channel, the Irish Sea, and the west coasts of Ireland and Scotland. The two latter waves affect the motion within the Irish Sea, and their crests reach the St. George's Channel and the North Channel respectively almost simultaneously, namely, at 6 h. 25 m. G.M.T. on the days* of full and change of the moon. At this time it is low water, or nearly so, on the Lancashire coast. The amplitude of the southern wave varies from about 4½ feet on the Irish coast to 10½ feet on the Welsh coast along the line at which it enters the Irish Sea, while that of the northern wave is about 2 feet when it enters the North Channel. The southern undulation dominates the motion as far as the south of Scotland.

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* All the data used refer to these days.

The range of tide between high and low water is known with considerable accuracy along the coasts. Captain Beechey,* in his 'Report on the Tides of the Irish Sea,' gives a chart showing the lines of equal range over the whole area. These lines are almost equally spaced along any section drawn from the Irish to the Welsh coasts—a result which was *a priori* probable, and, as Mr. Taylor has shown,† can be inferred from simple dynamical reasoning.

In the same way, the time of high water, or establishment, is known accurately along the coast, but only approximately in other places. The chart of co-tidal lines used is that issued by the Admiralty. In the southern region of the Irish Sea the divergence between different charts of these lines is not very large, but near the North Channel some (or all) are quite unreliable.‡ For this reason these first order calculations have not been applied to places further north than the Calf of Man.

Captain Beechey's Report shows also the lines of the in-going or flood stream. The out-going stream does not differ materially from the reverse of this except that in the St. George's Channel it presses rather more towards the Irish coast.

Throughout the whole of the area the maximum surface current occurs at 8 h. 22 m., so that τ is a constant. The magnitudes of this current at certain points on a number of sections are given in the Admiralty publication, 'The Tides and Tidal Streams of the British Islands.'§ The following sections have been used for the present purpose:—

Section I. Tuskar Island (52° 12′ N., 6° 12′ W.) to St. David's Head (51° 54′ N., 5° 19′ W.).

Section II. Near the South End of Arklow Bank (52° 42′ N., 6° 0′ W.) to Bardsey Island (52° 45′ N., 4° 48′ W.).

Section III. Near Kish Lightship (53° 19½′ N., 5° 54′ W.) to North Stack, Holyhead (53° 19½′ N., 4° 41′ W.).

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* 'Phil. Trans.,' p. 105 (1848). This report is the basis of the most recent Admiralty publications.

† *Loc. cit.*, *ante*, p. 9.

‡ The wave moves very slowly through the North Channel, its rate of progress being about two miles per hour, while further south its motion is very rapid, and the space gradients of T are quite indeterminable. The variation in longitude throughout the region has been neglected.

§ The greater part of the description of the tidal motion is derived from this source.

Section IV. Rockabill (53° 36′ N., 6° 0′ W.) to the Calf of Man (54° 3′ N., 4° 50′ W.).

Section V. The Calf of Man to Skerries Lighthouse (53° 25′ N., 4° 36′ W.).

Each of these sections is divided into five approximately equal portions, and the maximum current at each of the points of division is recorded. These stations are denoted by I*a*, I*b*, ... II*a*, ... the numeral denoting the section on which the station is situated.

In addition to the above, detailed hourly observations are available for the following lightships within this region:—(A) South Arklow 52° 41′ N., 5° 56′ W.), (B) Cardigan Bay (52° 24′ N., 5° 0′ W.), (C) North Arklow (52° 54′ N., 5° 50′ W.), (D) Codling Bank (53° 4′ N., 5° 45′ W.), (E) Carnarvon Bay (53° 6′ N., 4° 44′ W.), (F) Kish (53° 19½′ N., 5° 54′ W.). These stations are denoted by the letters A ... F.

The results of observations and of measurements made on the charts are shown in Table I. For convenience the units in common use are retained; heights are measured in feet, distances in nautical miles, and speeds in knots. A blank in a column indicates that the gradient is relatively small and uncertain. For the purpose of calculation its value is taken as zero.

Comparison of First-order Theory with Observations.

Equation (6) determines w when W, H_x, H_y are known, but the calculation is simplified if the first two equations (4) are used, $\sigma(\mathrm{T}-\tau)$ being considered an unknown variable, θ. Writing for the semi-diurnal tide $\sigma = 2\Omega$ and $\omega = \Omega \sin\lambda$, where Ω is the earth's axial rotation and λ is the latitude, we have from (4)

$$\left.\begin{aligned} g\mathrm{H}_x &= 2\Omega\,(\mathrm{W}+w\sin\lambda)\sin\theta \\ -g\mathrm{H}_y &= 2\Omega\,(\mathrm{W}\sin\lambda+w)\cos\theta \end{aligned}\right\}, \qquad (8)$$

from which by eliminating w

$$g\mathrm{W}^{-1}(\mathrm{H}_x\cos\theta+\mathrm{H}_y\sin\lambda\sin\theta) = \Omega\cos^2\lambda\sin 2\theta. \qquad (9)$$

From this equation θ is found; either of the equations (8) then determines w.

As results are required at each of the observing stations a graphical solution is most convenient, and owing to the small variation of $\cos^2\lambda$ over the southern part of the Irish Sea one graph of the function on the right of equation (9) is sufficient.

Table I.—Observational Data.

Station.	Feet.	Feet per nautical mile.			G.M.T.	Hours per nautical mile.	Degrees.	Knots.	
	H.	H′.	H_x.	H_y.	T.	T′.	χ.	W.	w.
					h. m.				
I*a*	5·0	0·14	−0·09	−0·12	6 20	0·06	195	3	
I*b*	5·8	0·13	−0·06	−0·12	6 50	0·04	165	2·8	
I*c*	7·0	0·09	—	−0·09	6 5	0·04	30	2·5	
I*d*	7·5	0·16	—	−0·12	6 15	0·05	60	4	
A	1·8	0·23	0·16	−0·19	7 50	0·07	80	3·2	−0·3
B	6·5	0·06	—	−0·06	6 40	0·02	80	2·5	−0·4
II*a*	2·8	0·17	0·10	−0·12	7 55	0·09	80	3·6	
II*b*	4·3	0·16	0·08	−0·13	7 40	0·08	80	3·2	
II*c*	6·0	0·14	0·06	−0·14	7 30	0·08	90	3·2	
II*d*	7·5	0·10	—	−0·10	7 35	0·09	90	3	
C	3·5	0·14	0·14	−0·07	8 50	0·14	85	3·8	0
D	5·0	0·14	0·12	−0·07	9 30	0·14	90	3·5	−0·3
E	7·9	0·08	—	—	8 55	0·07	0	2·2	0
F	6·5	0·10	0·06	−0·07	11 0	0·13	105	2·2	0
III*a*	7·0	0·10	0·08	−0·06	10 20	0·16	115	2	
III*b*	7·5	0·08	0·06	−0·05	9 30	0·06	105	2·5	
III*c*	8·3	0·07	0·05	−0·05	9 20	0·05	30	2·5	
III*d*	9·0	0·08	0·06	−0·05	10 0	0·10	0	3·5	
IV*a*	7·5	0·08	0·04	0·07	11 20	?	140	1	
IV*b*	8·3	0·07	0·05	−0·05	11 0	0·16	110	0·5	
IV*c*	9·1	0·05	0·04	—	10 50	0·13	110	1·2	
IV*d*	9·6	0·11	0·06	—	11 0	0·07	135	1·2	
V*a*	10·3	0·10	0·09	—	10 50	0·06	250	2·5	
V*b*	10·4	0·12	0·12	—	10 30	0·04	310	1·5	
V*c*	10·3	0·14	0·14	—	10 15	0·03	340	2·0	
V*d*	10·2	0·14	0·14	—	10 15	0·03	0	3·0	

With the units used, equation (9) becomes

$$W^{-1}\sqrt{(H_x^2+0{\cdot}64H_y^2)}\sin(\alpha-\theta) = 0{\cdot}0084\sin 2\theta, \qquad (10)$$

where $$H_x \operatorname{cosec}\alpha = -0{\cdot}8H_y\sec\alpha = \sqrt{(H_x^2+0{\cdot}64H_y^2)},$$

so that if it is assumed that $W+w\sin\lambda$ and $W\sin\lambda+w$ are both positive,* the angles α, θ, are in the same quadrant.

The second column of Table II gives the values of θ obtained by graphical solution of (10), and the third column the values of w deduced from (8). The results are of the right order of magnitude and with one exception† are numerically less than the corresponding values of W. The last column of Table I contains the values of w deduced from the hodographs of the observed

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* W is positive, and usually w/W is of order less than $\frac{1}{8}$.

† For Station IV*a*. As might have been foreseen, the general agreement is closer in the south, where the tidal range is smaller.

Table II.

Station.	θ (degrees).	w (knots).	W + w (knots).		
			Calculated.	Observed.	W (observed). w (calculated).
I*a*	−33	0·7	2·2	—	3·7
I*b*	−23	0·6	2·3	—	3·4
I*c*	0	0·0	2·3	—	2·5
I*d*	0	−0·6	4·0	—	3·4
A	39	2·7	3·5	2·9	
B	0	−0·7	1·3	2·1	
II*a*	34	0·2	3·7	—	3·8
II*b*	28	1·0	4·0	—	4·2
II*c*	20	0·6	4·6	—	3·8
II*d*	0	−0·3	5·2	—	2·7
C	59	−0·1	4·2	3·8	
D	52	−0·4	6·1	3·1	
E	?	?	3·7	2·2	
F	35	0·0	5·7	2·2	
III*a*	49	0·4	7·6	—	2·4
III*b*	41	−0·6	3·3	—	1·9
III*c*	32	−0·8	3·1	—	1·7
III*d*	35	−1·5	5·7	—	2·0
IV*a*	−43	1·3	?	—	2·3
IV*b*	45	1·1	8·4	—	1·6
IV*c*	90	−0·5	7·9	—	0·7
IV*d*	90	0·2	5·2	—	1·4
V*a*	90	−0·5	2·7	—	2·0
V*b*	90	1·5	1·7	—	3·0
V*c*	90	1·4	1·0	—	3·4
V*d*	90	0·2	2·5	—	3·2

velocities at the lightships A ... F, and except in one case the calculated results are in close agreement.

Since $\theta = \sigma(\mathrm{T}-\tau)$ and $\tau =$ 8 h. 22 m., the values of T can be deduced from those of θ. No great accuracy is possible, but the values of θ indicate (except in the case of IV*a*) the progress of the tidal wave northward.

With the same units as before (5) becomes

$$\left.\begin{aligned} \mathrm{W}+w &= 12\sqrt{(\mathrm{H}'^2+1{\cdot}1\,\mathrm{HH}'\mathrm{T}'\sin\chi+0{\cdot}27\,\mathrm{H}^2\mathrm{T}'^2)} \\ \mathrm{W}-w &= 110\sqrt{(\mathrm{H}'^2-1{\cdot}1\,\mathrm{HH}'\mathrm{T}'\sin\chi+0{\cdot}27\,\mathrm{H}^2\mathrm{T}'^2)} \end{aligned}\right\}, \tag{11}$$

but as the numerical value of the second term in the bracket is found to be almost equal to the sum of the other two terms, the second of these equations is quite useless. The fourth column of Table II gives the values of W + w calculated from (11), and the following column such values of W + w as can be deduced directly from observations. For the stations on the five sections w is not known from observation, and its calculated value has been used. The values of $W+w$ so obtained are given in the last column.

The agreement between the observed and the calculated results is not so good as before. The variables here used for the first time depend entirely on the cotidal lines, so that the closeness of the agreement may be taken as a measure of their accuracy. It should, however, be mentioned that a glance at a chart will often suggest reasons for the more marked discrepancies due to the configuration of the coast or of the bed of the sea.

Integrating the first of equations (7) along a line of flood stream from A to B we obtain

$$W_A - W_B = \sigma \int_A^B \frac{H}{h} \sin \sigma (T - \tau)\, ds, \tag{12}$$

although no great accuracy is to be expected as the variation of h has been neglected previously.

The value of the integral can be found by graphical methods, the depths, h, being taken from a chart.* The ranges of integration are taken between the sectional lines I, II, III, V. The calculation has been carried out for the lines of flood-stream which cut Section I in the points (52° 5′ N., 5° 54′ W.) and (52° 2′ N., 5° 45′ W.). The Table shows the changes of W, as before measured in knots.

Range of integration.	First stream line.		Second stream line.	
	Calculated.	Observed.	Calculated.	Observed.
Section I to Section II	0·3	0·3	0·4	0·5
" II " III	0·0	−0·7	−0·1	−0·6
" III " V	−0·6	−1·0	−0·5	−0·5

Between Sections I and II and between Sections III and V the agreement is quite satisfactory. Between Sections II and III, the integrand changes from positive to negative, so that a small error in T will have an important effect on the integral, and very accurate results should not be expected.

From equation (7) it is seen that the value $T = \tau = 8$ h. 22 m. makes W a maximum as far as variation with x is concerned. This result agrees well with observation.

The corresponding equation for the variation of w cannot usefully be evaluated in a similar manner as there are no sets of observed values of w with which to compare the results. Moreover, the change of h is much larger here than in the previous case. It is, however, interesting to note that $\partial w/\partial y$ is positive in the southern portion of the Irish Sea, so that there should be a decrease in the value of w on crossing towards the Welsh coast in a direction at right angles to the lines of flood stream. Sections I, II, III, V are very nearly in this direction, and with two exceptions, the calculated values of w given in third column of Table II satisfy the required conditions.

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* Admiralty Chart 1824A has been used.

A Second Approximation to Tidal Relations in a Flat Rotating Sea.

Using the same axes as before and taking the origin at the bottom of the sea, let the components of the relative velocities of the water referred to these axes be $U+U'$, $V+V'$, W', where accented quantities are small compared with unaccented. Then if we retain terms of the two highest orders only the equations of motion may be written

$$\left.\begin{aligned} \frac{\partial}{\partial t}(U+U')+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}-2\omega(V+V')-\nu\Delta(U+U') & \\ = -\frac{\partial}{\partial x}(\Phi+\Phi') & \\ \frac{\partial}{\partial t}(V+V')+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+2\omega(U+U')-\nu\Delta(V+V') & \\ = -\frac{\partial}{\partial y}(\Phi+\Phi') & \\ \frac{\partial W'}{\partial t}-\nu\Delta W' = -\frac{\partial}{\partial z}(\Phi+\Phi') & \end{aligned}\right\}, \tag{13}$$

where ν is the coefficient of viscosity and the space gradients of Φ' are small compared with those of Φ.

The first approximation to these equations is

$$\left.\begin{aligned} \frac{\partial U}{\partial t}-2\omega V-\nu\frac{\partial^2 U}{\partial z^2} &= -\frac{\partial \Phi}{\partial x} \\ \frac{\partial V}{\partial t}+2\omega U-\nu\frac{\partial^2 V}{\partial z^2} &= -\frac{\partial \Phi}{\partial y} \end{aligned}\right\}. \tag{14}$$

The solution in deep water represents motion vanishing at the bottom; in terms of complex velocities, $\mathbf{U}$, $\mathbf{V}$ it is given* by

$$\mathbf{U}\pm\iota\mathbf{V} = (\mathbf{U}_0\pm\iota\mathbf{V}_0)\left[1-\exp\left\{-\begin{matrix}\alpha\\ \beta\end{matrix}\right\}(1+\iota)z\right], \tag{15}$$

where $\mathbf{U}_0$, $\mathbf{V}_0$ are the surface values of $\mathbf{U}$, $\mathbf{V}$, and $2\nu\alpha^2=\sigma+2\omega$, $2\nu\beta^2=\sigma-2\omega$, the motion being simple harmonic with the speed σ of the tide under consideration. In obtaining these results, we have replaced $\tanh \alpha h$ and $\tanh \beta h$ by unity; but, as $1/\alpha = 12$ cm. and $1/\beta = 25$ cm., the error so introduced is excessively small in deep water.

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* R. O. Street, *loc. cit.*, *ante.* The rejection of the other terms depending on the viscosity is there justified. It is assumed that ω is not exceedingly small compared with σ and Ω.

If we now assume that we can replace $\Delta(\mathrm{U}', \mathrm{V}')$ by $\partial^2(\mathrm{U}', \mathrm{V}')/\partial z^2$, the parts of (13) *which are independent of the time* are

$$\left.\begin{aligned} 2\omega v + \nu\frac{\partial^2 u}{\partial z^2} &= \tfrac{1}{2}\Sigma\left(\mathrm{U}_r\frac{\partial}{\partial x}+\mathrm{V}_r\frac{\partial}{\partial y}\right)\mathrm{U}_r + \frac{\partial\phi}{\partial x} \\ -2\omega u + \nu\frac{\partial^2 v}{\partial z^2} &= \tfrac{1}{2}\Sigma\left(\mathrm{U}_r\frac{\partial}{\partial x}+\mathrm{V}_r\frac{\partial}{\partial y}\right)\mathrm{V}_r + \frac{\partial\phi}{\partial y} \\ \nu\frac{\partial^2 w}{\partial z^2} &= \frac{\partial\phi}{\partial z} \end{aligned}\right\}, \tag{16}$$

where u, v, w, ϕ are the parts of $\mathrm{U}', \mathrm{V}', \mathrm{W}', \Phi'$, which are independent of the time, and we have written, as is clearly permissible from (14),

$$\mathrm{U} = \mathrm{U}_1\cos\sigma t + \mathrm{U}_2\sin\sigma t, \qquad \mathrm{V} = \mathrm{V}_1\cos\sigma t + \mathrm{V}_2\sin\sigma t. \tag{17}$$

The summation refers to the values $r = 1, 2$.

From the last of equations (16)

$$\phi + f(x, y) = \nu\frac{\partial w}{\partial z} = -\nu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right),$$

so that, to the degree of approximation already used, $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are independent of z.

Now let the real parts of $\mathbf{U}_0$, $\mathbf{V}_0$, which we may call the components of the surface currents, be

$$p_1\cos\sigma t + p_2\sin\sigma t, \qquad q_1\cos\sigma t + q_2\sin\sigma t. \tag{18}$$

Then from the first two of equations (16) after some reduction

$$\left(\nu\frac{\partial^2}{\partial z^2} - 2\iota\omega\right)(u+\iota v) = \left(\frac{\partial}{\partial x}+\iota\frac{\partial}{\partial y}\right)\phi + (\mathrm{S}+\iota\mathrm{T})\left[\mathrm{P}(\mathrm{S}+\iota\mathrm{T})+\mathrm{Q}(\mathrm{S}-\iota\mathrm{T})\right], \tag{19}$$

where

$$\left.\begin{aligned} 2\mathrm{S} &= 2 - e^{-\alpha z}\cos\alpha z - e^{-\beta z}\cos\beta z \\ 2\mathrm{T} &= e^{-\alpha z}\sin\alpha z - e^{-\beta z}\sin\beta z \\ 4\begin{Bmatrix}\mathrm{P}\\ \mathrm{Q}\end{Bmatrix} &= \Sigma\left[(p_r \pm \iota q_r)\left(\frac{\partial}{\partial x} \mp \iota\frac{\partial}{\partial y}\right)(p_r+\iota q_r)\right] \qquad r = 1,\ 2 \end{aligned}\right\}, \tag{20}$$

so that P, Q, are independent of z.

The right-hand side of equation (19) is thus seen to take the form

$$\mathrm{P}+\mathrm{Q}+\left(\frac{\partial}{\partial x}+\iota\frac{\partial}{\partial y}\right)\phi + \Sigma\,(l\mathrm{P}+m\mathrm{Q})\,e^{-(a\alpha+b\beta)z}$$

where the summation includes nine terms and a, b, l, m are numerical, the real parts of a, b being positive. Hence, if $\nu\gamma^2 = 2\iota\omega$, the solution of (19), with the boundary conditions u, v zero at the bottom $z = 0$, $\partial(u, v)/\partial z$ zero at $z = h$, is given by

$$u+\iota v = \frac{\iota}{2\omega}\left[\mathrm{P}+\mathrm{Q}+\frac{\partial\phi}{\partial x}+\iota\frac{\partial\phi}{\partial y}\right]\left[1-\frac{\cosh\gamma(h-z)}{\cosh\gamma h}\right]$$

$$+\Sigma\frac{(l\mathrm{P}+m\mathrm{Q})\left[e^{-\delta z}\cosh\gamma h+\delta\gamma^{-1}e^{-\delta h}\sinh\gamma z-\cosh\gamma(h-z)\right]}{\nu(\delta^2-\gamma^2)\cosh\gamma h}, \tag{21}$$

where $\delta = a\alpha+b\beta$ and δ^2 is not equal to γ^2 in any term.

For our future work we shall require to integrate u from 0 to h. Integrating (21), we find after some reduction as the first approximation,

$$\int_0^h (u+\iota v)\,dz = \frac{\iota h}{2\omega}\left[\mathrm{P}+\mathrm{Q}+\frac{\partial\phi}{\partial x}+\iota\frac{\partial\phi}{\partial y}\right].$$

Taking now the axis of x parallel to the direction of the flood stream at the surface, we obtain from this

$$\int_0^h u\,dz = -\frac{h}{4\omega}\left(w\frac{\partial w}{\partial y}-2\frac{\partial\phi}{\partial y}\right), \tag{22}$$

where w now denotes the minimum surface velocity at (x, y), as in equation (3).

The second equation in (16) applied to the bottom of the sea becomes

$$\left(\nu\frac{\partial^2 v}{\partial z^2}\right)_0 = \frac{\partial\phi}{\partial y},$$

so that near the bottom we have

$$v = \mathrm{C}z+\frac{z^2}{2\nu}\frac{\partial\phi}{\partial y}.$$

We may thus suppose that $\partial\phi/\partial y$, and so also $\nu\partial^2 v/\partial z^2$, is of the order $\nu w/h^2$, and therefore in deep water small compared with ωu. Hence with this special orientation of axes the second equation (16) at the surface takes the form

$$-2\omega u = \tfrac{1}{2}w\frac{\partial w}{\partial y}, \tag{23}$$

while (22) may be replaced by

$$\int_0^h u\,dz = -\frac{hw}{4\omega}\frac{\partial w}{\partial y}, \tag{24}$$

which, by (23) is equivalent to stating that the surface value of u is to this approximation equal to its mean value.

The Flow of Water across a Vertical Section.

Taking the section in the plane yOz, the rate at which water crosses per unit length is approximately

$$\int_0^{h+\zeta} \mathrm{U}\,dz + \int_0^h \mathrm{U}'\,dz.$$

If F is the mean value of this flow taken over a long period

$$\mathrm{F} = \text{Mean value of} \int_{h-\mathrm{H}}^{h+\zeta} \mathrm{U}\,dz + \int_0^h u\,dz. \tag{25}$$

The first integral for a section normal to the direction of the flood stream at the surface becomes

$$\tfrac{1}{2}\mathrm{HW}\cos\sigma(\mathrm{T}-\tau), \tag{26}$$

which by equation (4) is equivalent to

$$-\frac{g\mathrm{H}}{4\omega}\frac{\partial \mathrm{H}}{\partial y}, \tag{27}$$

on the assumption that $\sigma w/2\omega\mathrm{W}$ may be neglected.*

Using (24) in (25) we find

$$\mathrm{F} = -\frac{1}{4\omega}\left(hw\frac{\partial w}{\partial y} + g\mathrm{H}\frac{\partial \mathrm{H}}{\partial y}\right). \tag{28}$$

The Transfer of Energy across a Vertical Section.

Taking the same section, the rate of transfer of energy per unit length is

$$\int_0^{h+\zeta} \{g\rho(z-h) + \tfrac{1}{2}\rho\mathrm{Q}^2 + \Pi\}\,\mathrm{Q}_x\,dz, \tag{29}$$

where Q is the relative velocity of the water, Q_x its component along Ox, ρ the density of the water, supposed uniform, and Π the pressure. The mean level of the sea is taken as the zero of potential energy.

If l is the height of the water barometer at sea level, and

$$\Pi = g\rho(l+h+\zeta-z) + \mathrm{P}',$$

so that P′ is relatively small, (29) can to our approximation be replaced by

$$g\rho(l+\zeta)\int_0^{h+\zeta}\mathrm{U}\,dz + \int_0^h \{g\rho l\mathrm{U}' + \tfrac{1}{2}\rho\mathrm{U}(\mathrm{U}^2+\mathrm{V}^2) + \mathrm{UP}'\}\,dz.$$

The mean value of the first integral is†

$$\tfrac{1}{2}g\rho(h+l)\,\mathrm{HW}\cos\sigma(\mathrm{T}-\tau),$$

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* This transformation is made to eliminate T, and thereby to make possible the subsequent numerical work. The error introduced is probably less than that arising from the uncertainty as to the values of T.

† Neglecting l, this is equivalent to the expression given by Mr. G. I. Taylor (*loc. cit. ante*), for the total rate of transfer of energy.

which (on the same assumption as before) can be replaced by

$$\frac{-g^2\rho\,(h+l)\,\mathrm{H}}{4\omega}\frac{\partial \mathrm{H}}{\partial y}.$$

The mean value of the second integral is to our approximation

$$g\rho l\int_0^h u\,dz.$$

Thus if K_0 denotes the mean rate of transfer of energy per unit length of section

$$\mathrm{K}_0 = \frac{-g\rho}{4\omega}\left\{lhw\frac{\partial w}{\partial y}+g\,(h+l)\,\mathrm{H}\frac{\partial \mathrm{H}}{\partial y}\right\}. \tag{30}$$

If the zero of potential energy were taken at a height k above mean sea-level $(z-h)$ in (29) would be replaced by $(z+k-h)$, and in place of (30) we should have

$$\mathrm{K}_k = \frac{-g\rho}{4\omega}\left\{(l-k)\,hw\frac{\partial w}{\partial y}+g\,(l+h-k)\,\mathrm{H}\frac{\partial \mathrm{H}}{\partial y}\right\}, \tag{31}$$

so that from (28)

$$\mathrm{K}_0-\mathrm{K}_k = g\rho k\mathrm{F}.$$

If any two sections of the type considered are taken right across any channel, the integrals of F along them must be the same in the absence of any permanent change in the volume of water.* Hence in estimating the total transfer of energy into the region between them it is immaterial what zero of potential energy is used, provided the same is used for both. This is evident from physical considerations; it assumes that both mean sea-levels are on the same gravitational equipotential surface. Equation (31) takes its simplest form when $k = l$, when it may be written

$$\mathrm{K} = \frac{-g^2\rho h\mathrm{H}}{4\omega}\frac{\partial \mathrm{H}}{\partial y}. \tag{32}$$

Numerical Evaluation of the Transfer of Energy.

Sufficient data exist to enable the integrals to be evaluated across certain sections of the Irish Sea.

Section I.—The deviation of the maximum surface current from the normal to the section nowhere exceeds six degrees and may be neglected. The direction of integration† is towards the Irish Coast, and $\partial\mathrm{H}/\partial y$ is almost constant.

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* It is easy to show that the change due to rainfall is negligible in the case of the Irish Sea.

† This is determined by the direction of the flood stream. Actually, the integrals are taken between the twenty-fathom lines, which are usually within a mile of the coast.

To evaluate the integral, the section is divided into four equal parts and the mean of the five values of $h\mathrm{H}$ obtained by Simpson's rule. For the five points we have

$$h = 20,\ 49,\ 53,\ 51,\ 20 \text{ fathoms},$$
$$\mathrm{H} = 8{\cdot}4,\ 7{\cdot}1,\ 6{\cdot}3,\ 5{\cdot}4,\ 4{\cdot}5 \text{ feet},$$

so that the mean of $h\mathrm{H}$ is 285. Taking $\rho = 1{\cdot}03$ we obtain from (32) for the mean northward transfer of energy

$$8{\cdot}1 \times 10^{17} \text{ ergs per second.} \tag{33}$$

In a similar manner for Sections II and III we find

$$8{\cdot}2 \times 10^{17} \text{ and } 6{\cdot}4 \times 10^{17} \text{ ergs per second.} \qquad (34), (35)$$

Section V is in a suitable direction and along it the total change in H does not exceed 6 inches, its values at the two ends being approximately equal. Since the values of $h\mathrm{H}$ are only slightly greater than on the previous sections, it follows that the value of the integral is relatively very small for this section, and to our order may be taken as zero.

For Section XII, Burrow Head (54° 41′ N., 4° 24′ W.) to Ayre Point (54° 25′ N., 4° 22′ W.). The mean rate of transfer is

$$0{\cdot}2 \times 10^{17} \text{ ergs per second,} \tag{36}$$

the direction being eastward.

For Section VIII, Muck Island (54° 51′ N., 5° 43′ W.) to Corsewall Point (55° 1′ N., 5° 9′ W.) it is northward and of amount

$$1{\cdot}1 \times 10^{17} \text{ ergs per second,} \tag{37}$$

while for Section VII, Tor Point (55° 12′ N., 6° 4′ W.) to the Mull of Cantyre (55° 18′ N., 5° 48′ W.), it is in a north-westerly direction and does not exceed

$$0{\cdot}3 \times 10^{17} \text{ ergs per second.} \tag{38}$$

The Transfer of Water.

The term in the expression (28) for F which depends on w prevents any very exact calculations, but estimates can be made using the calculated values of w given in Table II. For each of the Sections I, II, III a result in the neighbourhood of 10^{11} c.c. per second is obtained, the contributions from the two terms of the integral being of the same order of magnitude.* Sufficient information is not available to obtain reliable results for integrations across the North Channel.

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* To check the results, the first term in the expression for F has been evaluated directly from (26) without making use of (4), T being obtained from the chart of cotidal lines. For the three sections, the results $0{\cdot}4$, $0{\cdot}8$, $0{\cdot}9 \times 10^{11}$ c.c. per sec. were obtained.

Discussion of the Results.

The estimation of the transfer of energy is somewhat arbitrary, owing to the absence of any precise zero of potential energy. The integrals evaluated on the hypothesis that the zero level is at a height above mean sea level equal to the height of the water barometer are, from (35)—(38):

For the southern sections ...... $8\cdot1$, $8\cdot2$, $6\cdot4 \times 10^{17}$ C.G.S. units
For the northern sections ...... $1\cdot1$, $0\cdot3 \times 10^{17}$ C.G.S. units,

the last being a maximum result. In each case the mean transfer is northward.

Now, throughout the Irish Sea the mean sea level is not constant, there being a rise of about 2 feet from the south to the north.[*] A further correction must be introduced since Ordnance levels are not of gravitational equipotential surfaces. We shall probably be safe if we suppose that, referred to an equipotential surface, the variation of mean sea level in the Irish Sea does not exceed a metre, and in this limit we may include any correction due to variation of mean atmospheric pressure. Thus the *maximum* corrections to be applied to any of the integrals to refer it to the same level as that used for Section VII is 100 $g\rho$F, or, say, 10^{16} ergs per second. As this certainly falls within the limit of error, it may be ignored.

Subtracting the integrals for the extreme Sections I and VII, we find, as the mean rate of transfer of energy into the Irish Sea,

$$7\cdot8 \times 10^{17} \text{ ergs per second.}$$

Actually, this result must be too large, for both the current velocities and the tidal ranges have been taken at their spring-tide values. At neap tides H is reduced to about three-quarters of this value, and presumably the currents will be decreased in about the same ratio.[†] Hence, as we are dealing with products, it is probably a good approximation to say that the mean rate of transfer of energy into the Irish Sea is[‡]

$$6 \times 10^{17} \text{ ergs per second.}$$

A detailed examination of the integrals shows that about one-third of this energy is gained by the portion of the sea south of a line drawn west through Holyhead, while only a small fraction of it flows into the region east of a line through the Isle of Man.

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[*] See G. B. Airey, 'Phil. Trans.,' 1845, p. 1. I am indebted to the Ordnance Survey for this reference.

[†] The introduction of long periods is of course inconsistent with the hypothesis that H, w are independent of the time.

[‡] Mr. Taylor's estimate (correcting a slight error in his working) is $7\cdot2 \times 10''$ ergs per second.

The residual flow of water is of the order 10^{11} c.c. per second, or 3000 cubic kilom. per annum, allowing for the reduction due to the long period motion. As the total water-content of the Irish Sea is about 3000 cubic kilom., we can say that it empties itself about once a year through the North Channel.

This residual drift of water has been evaluated previously by M. Knudsen* as equal at least to a yearly emptying of the Irish Sea. Its actual existence has been demonstrated by Dr. Bassett† and others from observations of the temperature and the salinity of the water at different periods of the year. Assuming the drift to be spread uniformly over the whole section, the residual velocity would be about 2 cm. per second, or 1 mile per day, in St. George's Channel, and about twice as much in the narrowest part of the North Channel.

My thanks are due to Sir Joseph Larmor for much valuable criticism of this paper before its communication to the Society.

[*Note added, December* 8.—The question has been suggested by a referee how far the horizontal forces causing this residual drift might be compensated by a small permanent slope of the free surface. But the motion has been estimated from the actual observed surface levels, so that the effect in thus partially piling up the water is discounted. The obstruction of the North Channel shows that this piling effect must be considerable; it is only if the northern exit were wholly obstructed that all horizontal forces would be compensated by the change of level.]

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* "Some Remarks on the Currents in the North Sea and Adjacent Waters," 'Pubs. de Circonstance,' vol. 39 (Conseil Perm. Internat. pour l'Exploration de la Mer).

† 'Liverpool Biol. Soc. Trans.,' vol. 24, p. 148 (1910); see also vol. 17, p. 154 (1903).